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An optimal control approach to curved rods

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Abstract

In this paper, a new approach to the generalized Naghdi model for the deformation of three-dimensional curved rods is studied. The method is based on optimal control theory.

1 Introduction

The so-called *control variational method* was introduced in several recent papers by the authors (see [1, 10, 11, 13]), especially in connection with problems that are related to the theory of Kirchhoff–Love arches or to various models of plates. Also the full three-dimensional linear elasticity system has been studied in [13]. Much of this material can be found in the recent monograph [7].

The basic idea is to refine the standard variational approach from the theory of partial differential equations by using the tools of modern optimal control theory. This offers more flexibility and allows to derive new results, both from a theoretical and a numerical point of view.

It should be noted that already Pironneau and Glowinski (cf. [8, 9]) pointed out that optimal control methods can be successfully applied to the numerical solution of the biharmonic equation.

In this paper, we study via the control variational method the generalized Naghdi model for three-dimensional curved rods, which was introduced by Ignat, Sprekels, and Tiba in [5]. In the next section, we will recall this model, and in Section 3 we demonstrate how optimal control theory can be applied to its study.

We underline that there are several possible ways to do this, which argues for the flexibility of our method. We also mention the simplicity of the control approach: for instance, the state equation used by us can be solved explicitly.

Finally, let us note that other approaches to curved rods can be found in the papers [3] and [14].

2 The Naghdi model

Let $\bar{\theta} \in W^{2,\infty}(0,L)^3$, $L > 0$, be the parametrization of a three-dimensional Jordan curve, which will be the line of centroids of the curved rod, and let $\omega \subset \mathbb{R}^2$ be some two-dimensional domain, which is not necessarily simply connected. If $\bar{t}, \bar{n}, \bar{b}$ denote the local orthonormal frame associated at each point $x_3 \in [0, L]$ with the curve $\bar{\theta}$, we define the geometric transformation

$$F : \Omega = \omega \times]0, L[\to F(\Omega) = \hat{\Omega} \subset \mathbb{R}^3 ,$$

$$F(\bar{x}) = F(x_1, x_2, x_3) = \bar{\theta}(x_3) + x_1 \bar{n}(x_3) + x_2 \bar{b}(x_3) \quad \forall (x_1, x_2) \in \omega , \ \forall x_3 \in]0, L[. \tag{2.1}$$

In the following, we will need the Jacobian of F, $J = \nabla F$, and its inverse, $J(\bar{x})^{-1} = (h_{ij}(\bar{x}))_{i,j=\overline{1,3}}$. We have

$$J(\bar{x})^{-1} = \begin{pmatrix} n_1 - \dfrac{c\, t_1 x_2}{\det J(\bar{x})} & n_2 - \dfrac{c\, t_2 x_2}{\det J(\bar{x})} & n_3 - \dfrac{c\, t_3 x_2}{\det J(\bar{x})} \\ b_1 + \dfrac{c\, t_1 x_1}{\det J(\bar{x})} & b_2 + \dfrac{c\, t_2 x_1}{\det J(\bar{x})} & b_3 + \dfrac{c\, t_3 x_1}{\det J(\bar{x})} \\ \dfrac{t_1}{\det J(\bar{x})} & \dfrac{t_2}{\det J(\bar{x})} & \dfrac{t_3}{\det J(\bar{x})} \end{pmatrix} , \tag{2.2}$$

$$\det J(\bar{x}) = 1 - \beta\, x_1 - a\, x_2 . \tag{2.3}$$

The computations to verify (2.2), (2.3) are elementary and can be found in [7, Chapter 6]. Moreover, in [7] the construction of a local frame is given that differs from the classical Frenet or Darboux frames and requires just $C^1[0, L]^3$ regularity for $\bar{\theta}$. The coefficients $a, \beta, c \in L^\infty(0, L)$ appearing in (2.2), (2.3) are similar to the torsion and curvature known from classical differential geometry and may be obtained by the "equations of motion":

$$\begin{aligned} \bar{t}'(x_3) &= a(x_3)\, \bar{b}(x_3) + \beta(x_3)\, \bar{n}(x_3) , \\ \bar{b}'(x_3) &= -a(x_3)\, \bar{t}(x_3) + c(x_3)\, \bar{n}(x_3) , \\ \bar{n}'(x_3) &= -\beta(x_3)\, \bar{t}(x_3) - c(x_3)\, \bar{b}(x_3) . \end{aligned} \tag{2.4}$$

We also assume that the selection of axes in $\omega \subset \mathbb{R}^2$ is made in such a way that

$$0 = \int_\omega x_1\, dx_1\, dx_2 = \int_\omega x_2\, dx_1\, dx_2 = \int_\omega x_1\, x_2\, dx_1\, dx_2 , \tag{2.5}$$

which is usual in the literature on curved rods, see [6]. If the diameter of ω is sufficiently small, then $|x_1|, |x_2|$ are small, and since β and a are bounded on $[0, L]$, relation (2.3) shows that we may assume that

$$\det J(\bar{x}) \geq K > 0 \quad \forall\, \bar{x} \in \Omega . \tag{2.6}$$

Then it is known that $F : \Omega \to \hat{\Omega}$ is a one-to-one transformation, and this justifies the geometric definition of the curved rod $\hat{\Omega} \subset \mathbb{R}^3$, see [4, Thm. 3.1-1].

The generalized Naghdi model may be stated as the following variational equation:

$$
\begin{aligned}
B(\bar{y},\bar{v}) \;=\; & \tilde{\lambda} \int_\Omega \sum_{i,j=1}^{3} \Big[N_i(x_3)\, h_{1i}(\bar{x}) + B_i(x_3)\, h_{2i}(\bar{x}) + (\tau_i'(x_3) + x_1\, N_i'(x_3) \\
& \qquad + x_2\, B_i'(x_3))\, h_{3i}(\bar{x}) \Big] \, \Big[M_j(x_3)\, h_{1j}(\bar{x}) + D_j(x_3)\, h_{2j}(\bar{x}) \\
& \qquad + (\mu_j'(x_3) + x_1\, M_j'(x_3) + x_2\, D_j'(x_3))\, h_{3j}(\bar{x}) \Big] \; \det J(\bar{x})\, d\bar{x} \\
& + \tilde{\mu} \int_\Omega \sum_{i<j} \Big[N_i(x_3)\, h_{1j}(\bar{x}) + B_i(x_3)\, h_{2j}(\bar{x}) + (\tau_i'(x_3) + x_1\, N_i'(x_3) \\
& \qquad + x_2\, B_i'(x_3))\, h_{3j}(\bar{x}) + N_j(x_3)\, h_{1i}(\bar{x}) + B_j(x_3)\, h_{2i}(\bar{x}) \\
& \qquad + (\tau_j'(x_3) + x_1\, N_j'(x_3) + x_2\, B_j'(x_3))\, h_{3i}(\bar{x}) \Big] \\
& \qquad \cdot \Big[M_i(x_3)\, h_{1j}(\bar{x}) + D_i(x_3)\, h_{2j}(\bar{x}) + (\mu_i'(x_3) + x_1\, M_i'(x_3) \\
& \qquad + x_2\, D_i'(x_3))\, h_{3j}(\bar{x}) + M_j(x_3)\, h_{1i}(\bar{x}) + D_j(x_3)\, h_{2i}(\bar{x}) \\
& \qquad + (\mu_j'(x_3) + x_1\, M_j'(x_3) + x_2\, D_j'(x_3))\, h_{3i}(\bar{x}) \Big] \; \det J(\bar{x})\, d\bar{x} \\
& + 2\, \tilde{\mu} \int_\Omega \sum_{i=1}^{3} \Big[N_i(x_3)\, h_{1i}(\bar{x}) + B_i(x_3)\, h_{2i}(\bar{x}) + (\tau_i'(x_3) + x_1\, N_i'(x_3) \\
& \qquad + x_2\, B_i'(x_3))\, h_{3i}(\bar{x}) \Big] \, \Big[M_i(x_3) h_{1i}(\bar{x}) + D_i(x_3)\, h_{2i}(\bar{x}) \\
& \qquad + (\mu_i'(x_3) + \; x_1\, M_i'(x_3) + x_2\, D_i'(x_3))\, h_{3i}(\bar{x}) \Big] \; \det J(\bar{x})\, d\bar{x} \\
=\; & \sum_{\ell=1}^{3} \int_\Omega f_\ell(\bar{x}) (\mu_\ell(x_3) + x_1\, M_\ell(x_3) + x_2\, D_\ell(x_3)) \det J(\bar{x})\, d\bar{x} \,,
\end{aligned}
\tag{2.7}
$$

for any test functions $\bar{\mu} = (\mu_1, \mu_2, \mu_3)$, $\bar{M} = (M_1, M_2, M_3)$, $\bar{D} = (D_1, D_2, D_3)$ in $H_0^1(0,L)^3$. We have $\bar{v} = (\bar{\mu}, \bar{M}, \bar{D}) \in H_0^1(0,L)^9$, and $\bar{y} = (\tau_1, \tau_2, \tau_3, N_1, N_2, N_3, B_1, B_2, B_3) \in H_0^1(0,L)^9$ is the vector of the unknowns. The bilateral null conditions, given by the choice of the space $H_0^1(0,L)$, correspond to a clamped curved rod. The model (2.7) is deduced from the linear elasticity system under the assumption that the displacement $\bar{y}$ has the form

$$
\bar{y}(\hat{x}) \;=\; \bar{\tau}(x_3) \;+\; x_1\, \bar{N}(x_3) \;+\; x_2\, \bar{B}(x_3)\,, \quad \forall\, \hat{x} \in \hat{\Omega}\,,
\tag{2.8}
$$

with $\bar{x} = (x_1, x_2, x_3) = F^{-1}(\hat{x}) \in \Omega$. A similar form is imposed for the test functions.

It should be clear that $\bar{\tau}$ describes the translation of the points on the line of centroids, and the vectors $\bar{N} + \bar{n}$, $\bar{B} + \bar{b}$ reflect the deformation of the orthogonal frame in the cross section (which remains plane but not necessarily orthogonal to the tangent of the new centroid line, i.e., to $\bar{\theta}' + \bar{\tau}'$). This allows for shear and for length or volume changes after the deformation. The vector $\bar{f} = (f_1, f_2, f_3)$ represents the body forces acting on the curved rods, $\tilde{\lambda} > 0$, $\tilde{\mu} \geq 0$ are the Lamé coefficients that characterize the elastic material, and the model is valid only for small displacements.

The assumption (2.8), which is in fact a first-order approximation of the displacement, enters the category of *polynomial models* for curved rods and allows to obtain realistic results even for the corresponding shape optimization problems; see [2]. We have called such a model a *generalized Naghdi model* in view of a certain similarity with Naghdi models for shells (cf. [4, 12]).

3 The control problem

We do not explain "how and why" this particular problem has been chosen, and we underline that this choice is not the only possible one (see the last remark in this section). The basic property of the chosen optimal control problem is that it solves system (2.7) and has a simple structure.

We formulate it now:

$$
\begin{aligned}
&\mathrm{Min}\ \{\tilde{\lambda} \int_\Omega \sum_{i,j=1}^{3} U_{ii}(\bar{x})\, U_{jj}(\bar{x})\ \det J(\bar{x})\, d\bar{x} \,+\, \tilde{\mu} \int_\Omega \sum_{i<j} [U_{ij}(\bar{x}) \,+\, U_{ji}(\bar{x})]^2\ \det J(\bar{x})\, d\bar{x} \\
&+ 2\,\tilde{\mu} \int_\Omega \sum_{i=1}^{3} U_{ii}^2(\bar{x}) \det J(\bar{x})\, d\bar{x} \,-\, 2 \sum_{i=1}^{3} \int_\Omega f_i(\bar{x})\, [\tau_i(x_3) \,+\, x_1\, N_i(x_3) \\
&+ x_2\, B_i(x_3)]\ \det J(\bar{x})\, d\bar{x}\}
\end{aligned}
\tag{3.1}
$$

subject to the state system

$$
N_i(x_3)\, h_{1j}(\bar{x}) + B_i(x_2)\, h_{2j}(\bar{x}) + [\tau_i'(x_3) + x_1\, N_i'(x_3) + x_2\, B_i'(x_3)]\, h_{3j}(\bar{x}) = U_{ij}(\bar{x}) \ \text{ in } \Omega\,, \tag{3.2}
$$

$$
N_i(0) \,=\, B_i(0) \,=\, \tau_i(0) \,=\, 0\,, \quad i \,=\, \overline{1,3}\,, \tag{3.3}
$$

and to the control constraints

$$
U \,=\, \{U_{ij}\}_{i,j=\overline{1,3}} \in \mathcal{V} \subset L^2(\Omega)^9\,. \tag{3.4}
$$

Here, $\mathcal{V}$ is the closed linear subspace that is generated from all functions in $L^2(0,L)$ having zero mean value (i.e., their integral over $[0,L]$ vanishes), used on the position of τ_i', N_i', B_i', $i = \overline{1,3}$. Then N_i, B_i can easily be computed by simple integration, and one can form all the combinations indicated on the left side of (3.2) to generate $\mathcal{V}$.

The state equation (3.2) is an ordinary differential system in the variable $x_3 \in [0,L]$, while $(x_1,x_2) \in \omega$ appear as parameters. The constraint $\{U_{ij}\}_{i,j=\overline{1,3}} \in \mathcal{V}$ ensures that (3.2) has a unique solution that also satisfies

$$\tau_i(L) = N_i(L) = B_i(L) = 0, \quad i = \overline{1,3}. \tag{3.5}$$

One could impose (3.5) as a state constraint in the problem (3.1)–(3.3), but we prefer to impose the restriction (3.4), which is explicit and constructive.

Theorem 3.1 *The optimal control problem* (3.1)–(3.3) *has a unique optimal "pair"* $U^* = \{U^*_{ij}\}_{i,j=\overline{1,3}} \in \mathcal{V}$ *and* $[\tau_i^*, N_i^*, B_i^*]_{i=\overline{1,3}} \in H_0^1(0,L)^9$.

Proof. Let $\mathcal{L}(U)$ denote the cost functional (3.1). Then

$$\alpha\,\mathcal{L}(U) \ge \int_\Omega \sum_{i<j}[U_{ij}(\bar{x})+U_{ji}(\bar{x})]^2\,d\bar{x} + \int_\Omega \sum_{i=1}^{3} U_{ii}^2(\bar{x})\,d\bar{x} - \bar{c}\sum_{i=1}^{3}\int_\Omega^L [\tau_i^2+N_i^2+B_i^2]^{\frac{1}{2}}\,dx_3, \tag{3.6}$$

where $\alpha > 0$, $\bar{c} > 0$, are constants obtained from $\tilde{\lambda}, \tilde{\mu}, |f_i|^2_{L^2(\Omega)}$, (2.3), and simple binomial inequalities.

Using (3.2) in (3.6), and again some binomial inequalities, we obtain (where we denote $z_i := \tau_i' + x_1 N_i' + x_2 B_i'$):

$$\begin{aligned}
\alpha\,\mathcal{L}(U) \ge & \int_\Omega \sum_{i<j} [N_i\,h_{1j} + B_i\,h_{2j} + z_i\,h_{3j} + N_j\,h_{1i} + B_j\,h_{2i} + z_j\,h_{3i}]^2\;d\bar{x} \\
& + \int_\Omega \sum_{i=1}^{3} [N_i\,h_{1i} + B_i\,h_{2i} + z_i\,h_{3i}]^2\;d\bar{x} \;-\; \bar{c}\sum_{i=1}^{3}\int_\Omega^L \left[\tau_i^2 + N_i^2 + B_i^2\right]^{\frac{1}{2}}\;dx_3 \\
\ge & \int_\Omega \sum_{i<j}(z_i\,h_{3j} + z_j\,h_{3i})^2\,d\bar{x} \;+\; \int_\Omega \sum_{i=1}^{3} z_i^2\,h_{3i}^2\,d\bar{x} \;-\; \hat{c}\sum_{i=1}^{3}\int_0^L [N_i^2 + B_i^2]\,dx_3 \\
& - \bar{c}\sum_{i=1}^{3}\int_0^L [\tau_i^2 + N_i^2 + B_i^2]^{\frac{1}{2}}\,dx_3\,.
\end{aligned} \tag{3.7}$$

Here, in the structure of the constant $\hat{c} > 0$, we also use that $h_{ij} \in L^\infty(\Omega)$, $i,j = \overline{1,3}$. We apply in (3.7) the following identities:

$$\frac{1}{2}\sum_{i<j}(z_i\,h_{3j}+z_j\,h_{3i})^2 + \frac{3}{2}\sum_{i=1}^{3} z_i^2\,h_{3i}^2 \;=\; \frac{1}{2}\sum_{i=1}^{3} z_i^2 \sum_{j=1}^{3} h_{3j}^2 + \frac{1}{2}\sum_{i<j}(z_i\,h_{3i}+z_j\,h_{3j})^2, \tag{3.8}$$

$$\int_\Omega z_i^2\, d\bar{x} \;=\; \int_\Omega [\tau_i' + x_1\, N_i' + x_2\, B_i']^2\, d\bar{x}$$

$$= \int_0^L \int_\omega (\tau_i')^2\, d\bar{x} \;+\; 2\int_0^L \tau_i'\, N_i'\, dx_3 \int_\omega x_1\, dx_1\, dx_2 \;+\; 2\int_0^L \tau_i'\, B_i'\, dx_3 \int_\omega x_2\, dx_1\, dx_2$$

$$+ \int_0^L (N_i')^2\, dx_3 \int_\omega x_1^2\, dx_1\, dx_2 \;+\; \int_0^L (B_i')^2 dx_3 \int_\omega x_2^2\, dx_1\, dx_2$$

$$+2\int_0^L N_i'\, B_i'\, dx_3 \int_\omega x_1\, x_2\, dx_1\, dx_2 \;\geq\; \tilde{c}\int_0^L \left[(\tau_i')^2 + (N_i')^2 + (B_i')^2\right]\, dx_3\,, \quad (3.9)$$

with $\tilde{c} := \min\,\{\mathrm{meas}(\omega)\,,\; \int_\omega x_1^2\, dx_1\, dx_2\,,\; \int_\omega x_2^2\, dx_1\, dx_2\}$,

$$\sum_{j=1}^{3} h_{3j}^2 \;=\; \frac{1}{\det J(\bar{x})}\sum_{j=1}^{3} t_j^2 \;=\; \frac{1}{\det J(\bar{x})} \;\geq\; \check{c} \;>\; 0 \quad \text{in}\ \Omega\,. \qquad (3.10)$$

Relations (3.9), (3.10) are a consequence of (2.5), respectively, of (2.2), (2.3). From (3.7)–(3.10) we can infer that

$$\alpha\,\mathcal{L}(U) \;+\; \bar{c}\sum_{i=1}^{3}\int_0^L [\tau_i^2 + N_i^2 + B_i^2]^{\frac{1}{2}}\, dx_3$$

$$\geq\; \hat{c}\sum_{i=1}^{3}\left[|\tau_i|^2_{H_0^1(0,L)} \;+\; |N_i|^2_{H_0^1(0,L)} \;+\; |B_i|^2_{H_0^1(0,L)}\right]$$

$$-\,\hat{C}\sum_{i=1}^{3}\left[|N_i|^2_{L^2(0,L)} \;+\; |B_i|^2_{L^2(0,L)}\right]^2\,, \qquad (3.11)$$

where $\hat{c}, \hat{C}, \bar{c}$ are some positive constants that do not depend on $\tau_i, N_i, B_i, U_{ij},\;\; i,j = \overline{1,3}$.

Lemma 3.2 *There is some $\delta > 0$ such that*

$$\alpha\,\mathcal{L}(U) \;+\; \bar{c}\sum_{i=1}^{3}\int_0^L \left[\tau_i^2 + N_i^2 + B_i^2\right]^{\frac{1}{2}}\; dx_3$$

$$\geq\; \delta\sum_{i=1}^{3}\left[|\tau_i|^2_{H_0^1(0,L)} \;+\; |N_i|^2_{H_0^1(0,L)} \;+\; |B_i|^2_{H_0^1(0,L)}\right] \qquad (3.12)$$

$\forall\; \tau_i, N_i, B_i \in H_0^1(0,L)\,,\;\; i = \overline{1,3}$*, obtained by* (3.2) *from any* $\{U_{ij}\}_{i,j=\overline{1,3}} \in \mathcal{V}$.

Proof. Owing to (3.6),

$$\alpha\,\mathcal{L}(U) \,+\, \bar{c}\sum_{i=1}^{3}\int_{0}^{L}\left[\tau_i^2 + N_i^2 + B_i^2\right]^{\frac{1}{2}}\,dx_3 \,\geq\, 0\,.$$

Assume that (3.12) is false. Then, to any $\varepsilon > 0$ there are some $U^\varepsilon = \{U^\varepsilon_{ij}\}_{i,j=\overline{1,3}} \in \mathcal{V}$, and $\{\tau_i^\varepsilon, N_i^\varepsilon, B_i^\varepsilon\}_{i=\overline{1,3}} \neq 0$ associated with it by (3.2), such that

$$\begin{aligned}
&\varepsilon\sum_{i=1}^{3}\left[|\tau_i^\varepsilon|^2_{H_0^1(0,L)} + |N_i^\varepsilon|^2_{H_0^1(0,L)} + |B_i^\varepsilon|^2_{H_0^1(0,L)}\right]\\
&\geq\, \alpha\,\mathcal{L}(U^\varepsilon) \,+\, \bar{c}\sum_{i=1}^{3}\int_{0}^{L}\left[(\tau_i^\varepsilon)^2 + (N_i^\varepsilon)^2 + (B_i^\varepsilon)^2\right]^{\frac{1}{2}}\,dx_3\\
&\geq\, \int_\Omega\sum_{i<j}\left[U^\varepsilon_{ij}(\bar{x}) + U^\varepsilon_{ji}(\bar{x})\right]^2\,d\bar{x} \,+\, \int_\Omega\sum_{i=1}^{3}(U^\varepsilon_{ii})^2\,d\bar{x} \,\geq\, 0\,.
\end{aligned}\tag{3.13}$$

Notice that we may assume

$$\sum_{i=1}^{3}\left[|\tau_i^\varepsilon|^2_{H_0^1(0,L)} + |N_i^\varepsilon|^2_{H_0^1(0,L)} + |B_i^\varepsilon|^2_{H_0^1(0,L)}\right] \,=\, 1\,,$$

by scaling with the square root of this factor (if it differs from unity) in the equation (3.2) and in the first and last term of (3.13). Then we may assume that

$$\begin{gathered}
\tau_i^\varepsilon \,\to\, \tau_i\,, \quad N_i^\varepsilon \,\to\, N_i\,, \quad B_i^\varepsilon \,\to\, B_i\,, \quad i = \overline{1,3}\,,\\
\text{weakly in } H_0^1(0,L) \text{ and strongly in } L^2(0,L)\,.
\end{gathered}$$

Moreover, by virtue of the state equation (3.2), we also see that $U^\varepsilon_{ij} \to U_{ij}$ weakly in $L^2(\Omega)$, on a subsequence. Also, $\{\tau_i, N_i, B_i\}_{i=\overline{1,3}}$ satisfy together with $\{U_{ij}\}_{i,j=\overline{1,3}}$ equation (3.2), while $\{U_{ij}\} \in \mathcal{V}$.

Passing to the limit in (3.13), we find that

$$\int_\Omega\sum_{i<j}\left[U_{ij}(\bar{x}) + U_{ji}(\bar{x})\right]^2\,d\bar{x} \,+\, \int_\Omega\sum_{i=1}^{3}(U_{ii})^2\,dx \,=\, 0\,. \tag{3.14}$$

From (3.14) it follows that

$$N_i\,h_{1i} \,+\, B_i\,h_{2i} \,+\, z_i\,h_{3i} \,=\, 0\,, \quad i \,=\, \overline{1,3}\,, \tag{3.15}$$

$$N_i\,h_{1j} \,+\, B_i\,h_{2j} \,+\, z_i\,h_{3j} \,+\, N_j\,h_{1i} \,+\, B_j\,h_{2i} \,+\, z_j\,h_{3i} \,=\, 0\,, \quad i \neq j\,. \tag{3.16}$$

Now fix $j = j_0$ in (3.16), and let i_1, i_2 be the two possible choices of indices i satisfying the condition in (3.16).

We multiply (3.15), written for $i = i_1$ or i_2, by h_{3j_0}, and subtract the result from (3.16) with $j = j_0$, multiplied by h_{3i_1}, respectively, h_{3i_2}. Adding the results to the relation (3.15), written for $i = j_0$ and multiplied by h_{3j_0}, we see that

$$z_{j_0} \sum_{i=1}^{3} h_{3i}^2 \;=\; \widetilde{\Gamma}_{j_0}(\bar{N}, \bar{B}), \quad j_0 \;=\; \overline{1,3}, \tag{3.17}$$

where $\widetilde{\Gamma}_j$ is some linear expression of $\bar{N}, \bar{B}$. By (2.2), we obtain from (3.17) that

$$\tau_i' \,+\, x_1 N_i' \,+\, x_2 B_i' \;=\; \Gamma_i(\bar{N}, \bar{B}), \quad i \;=\; \overline{1,3}, \tag{3.18}$$

where, again, Γ_i is linear in $\bar{N}, \bar{B}$.

Giving $(x_1, x_2) \in \omega$ several independent values, and taking into account (3.3), we conclude that all the limit points $\{\tau_i, N_i, B_i\}_{i=\overline{1,3}}$, $\{U_{ij}\}_{i,j=\overline{1,3}}$, are identically zero in their domains of definition.

Since the convergence of $\{\tau_i^\varepsilon, N_i^\varepsilon, B_i^\varepsilon\}_{i=\overline{1,3}}$ is strong in $L^2(0,L)^9$, we can pass to the limit in (3.11), combined with the first inequality in (3.13), to arrive at the contradiction

$$0 \,\geq\, \hat{c} \,>\, 0\,.$$

This ends the proof of **Lemma 3.2**. ■

Proof of Theorem 3.1 (continued): Let $\{U_{ij}^n\}_{i,j=\overline{1,3}}$ and $\{\tau_i^n, N_i^n, B_i^n\}$ be a minimizing sequence in $\mathcal{V} \times L^2(0,L)^9$ for the problem (3.1)–(3.4). Clearly, $\mathcal{L}(U^n)$ is majorized from above, and inequality (3.12) shows that $\{\tau_i^n, N_i^n, B_i^n\}_{i=\overline{1,3}}$ is bounded in $H_0^1(0,L)^9$. Consequently, by (3.2), $\{U_{ij}^n\}_{i,j=\overline{1,3}}$ is bounded in $L^2(\Omega)^9$. Let $\{\tau_i^*, N_i^*, B_i^*\}_{i=\overline{1,3}}$ and $\{U_{ij}^*\}_{i,j=\overline{1,3}}$ denote their respective weak limits, on a subsequence. Clearly $\{U_{ij}^*\}_{i,j=\overline{1,3}} \in \mathcal{V}$, since $\mathcal{V}$ is a closed linear subspace. Now, we can pass to the limit in (3.2) and use the weak lower semicontinuity of (3.1) to conclude that $\{\tau_i^*, N_i^*, B_i^*\}_{i=\overline{1,3}}$ and $\{U_{ij}^*\}_{i,j=\overline{1,3}}$ indeed give an optimal pair for the problem (3.1)–(3.4).

The uniqueness is an automatic consequence of the next result and of (3.2). ■

Theorem 3.3 *The optimal state $\{\tau_i^*, N_i^*, B_i^*\}_{i=\overline{1,3}}$ is the unique solution to the system* (2.7) *that governs the generalized Naghdi model.*

Proof. For any $\{V_{ij}\}_{i,j=\overline{1,3}} \in \mathcal{V}$, we define the system in variations by

$$\begin{aligned} & M_i(x_3)\, h_{1j}(\bar{x}) \,+\, D_i(x_3)\, h_{2j}(\bar{x}) \,+\, [\mu_i'(x_3) \,+\, x_1 M_i'(x_3) \\ & + x_2 D_i'(x_3)]\, h_{3j}(\bar{x}) \;=\; V_{ij}(\bar{x}), \quad i,j \;=\; \overline{1,3}, \end{aligned} \tag{3.19}$$

$$M_i(0) \;=\; D_i(0) \;=\; \mu_i(0) \;=\; 0, \quad i \;=\; \overline{1,3}. \tag{3.20}$$

Next, we perform admissible variations about the optimal pair, given by

$$\{\tau_i^*, N_i^*, B_i^*\}_{i=\overline{1,3}} + \lambda\{\mu_i, M_i, D_i\}_{i=\overline{1,3}}, \text{ and } \{U_{ij}^*\}_{i,j=\overline{1,3}} + \lambda\{V_{ij}\}_{i,j=\overline{1,3}}, \quad \lambda \in \mathbb{R}.$$

Subtracting the associated costs, dividing by $\lambda > 0$ or $\lambda < 0$, and taking the limits as $\lambda \to 0$, the minimum property of $\{U_{ij}^*\}_{i,j=\overline{1,3}}$ yields the associated Euler equation,

$$\begin{aligned} 0 \;=\; \tilde{\lambda} \int_\Omega \sum_{i,j=1}^{3} \left[U_{ii}^* V_{jj} + U_{jj}^* V_{ii}\right] \det J\, d\bar{x} \;+\; 2\,\tilde{\mu} \int_\Omega \sum_{i<j} \left[U_{ij}^* + U_{ji}^*\right] \left[V_{ij} + V_{ji}\right] \det J\, d\bar{x} \\ + 4\,\tilde{\mu} \int_\Omega \sum_{i=1}^{3} U_{ii}^* V_{ii} \det J\, d\bar{x} \;-\; 2 \int_\Omega \sum_{i=1}^{3} f_i \left[\mu_i + x_1 M_i + x_2 D_i\right] \det J\, d\bar{x}\,, \\ \forall\ \{V_{ij}\}_{i,j=1,3} \in \mathcal{V}\,. \end{aligned} \tag{3.21}$$

If the V_{ij} are replaced as in (3.19), (3.20), and the U_{ij}^* are replaced as in (3.2), then a simple computation shows that (3.21) becomes (2.7), which is known to have a unique solution. ■

Proposition 3.4 *If $\{U_{ij}^*\}$ is known, then $\{\tau_i^*, N_i^*, B_i^*\}_{i=\overline{1,3}}$ can be computed explicitly.*

Proof. Starting from (2.2), one can check the following orthogonality-type relations

$$\sum_{j=1}^{3} h_{1j}\, b_j \;=\; 0\,, \quad \sum_{j=1}^{3} h_{3j}\, b_j \;=\; 0\,, \quad \sum_{j=1}^{3} h_{2j}\, b_j \;=\; 1\,, \tag{3.22}$$

$$\sum_{j=1}^{3} h_{1j}\, n_j \;=\; 1\,, \quad \sum_{j=1}^{3} h_{2j}\, n_j \;=\; 0\,, \quad \sum_{j=1}^{3} h_{3j}\, n_j \;=\; 0\,, \tag{3.23}$$

$$\sum_{j=1}^{3} h_{1j}\, t_j \;=\; -\frac{c\, x_2}{\det J(\bar{x})}\,, \quad \sum_{j=1}^{3} h_{2j}\, t_j \;=\; \frac{c\, x_1}{\det J(\bar{x})}\,, \quad \sum_{j=1}^{3} h_{3j}\, t_j \;=\; \frac{1}{\det J(\bar{x})}\,. \tag{3.24}$$

Consequently, multiplying the equations (3.2) containing N_i^*, B_i^*, z_i^* by n_j (respectively, b_j, t_j) and adding for $j = \overline{1,3}$, we obtain from (3.22)–(3.24) the relations

$$B_i^* \;=\; \sum_{j=1}^{3} U_{ij}^*\, b_j\,, \quad i \;=\; \overline{1,3}\,, \tag{3.25}$$

$$N_i^* \;=\; \sum_{j=1}^{3} U_{ij}^*\, n_j\,, \quad i \;=\; \overline{1,3}\,, \tag{3.26}$$

$$(\tau_i^*)' + x_1 (N_i^*)' + x_2 (B_i^*)' = \sum_{j=1}^{3} U_{ij}^* t_j \det J(\bar{x}) + \sum_{j=1}^{3} U_{ij}^* n_j c x_2 - \sum_{j=1}^{3} U_{ij}^* b_j c x_1, \quad i = \overline{1,3}. \tag{3.27}$$

Thus, integrating (3.27) over $[0, x_3]$, and subtracting (3.25), (3.26), we get the explicit formula for $\{\tau_i^*\}_{i=\overline{1,3}}$, which completes the ones given by (3.25), (3.26). ■

Remark: Let us denote by $L_i(U_{ij})$ the right-hand side of (3.27). Then we can perform the following substitution in (3.1):

$$\sum_{i=1}^{3} \int_\Omega f_i [\tau_i + x_1 N_i + x_2 B_i] \det J \, d\bar{x}$$

$$= -\sum_{i=1}^{3} \int_\Omega [\tau_i' + x_1 B_i' + x_2 N_i'] \int_0^{x_3} f_i(x_1, x_2, \rho) \det J(x_1, x_2, \rho) \, d\rho \, d\bar{x}$$

$$= -\sum_{i=1}^{3} \int_\Omega L_i(U_{ij}) \int_0^{x_3} f_i(x_1, x_2, \rho) \det J(x_1, x_2, \rho) \, d\rho \, d\bar{x} .$$

In this way, the optimal control problem (3.1)–(3.4) can be transformed into a mathematical programming problem defined on $\mathcal{V} \subset L^2(\Omega)^9$, since the state is completely eliminated from the cost. However, one has to solve (3.2), or use (3.25)–(3.27), to compute the solution of (2.7), and we recommend to solve (3.1)–(3.4) directly, which is closer to the main problem given by (2.7).

Proposition 3.5 *The directional derivative of $\mathcal{L}(U_{ij})$, $i, j = \overline{1,3}$, in the direction $\{V_{ij}\}_{i,j=\overline{1,3}} \in \mathcal{V}$ is given by*

$$\langle \nabla \mathcal{L}(U_{ij}), \{V_{ij}\} \rangle = \tilde{\lambda} \int_\Omega \sum_{i,j=1}^{3} [U_{ii} V_{jj} + U_{jj} V_{ii}] \det J \, d\bar{x}$$

$$+ 2 \tilde{\mu} \int_\Omega \sum_{i<j} [U_{ij} + U_{ji}] \, [V_{ij} + V_{ji}] \det J \, d\bar{x} + 4 \tilde{\mu} \int_\Omega \sum_{i=1}^{3} U_{ii} V_{ii} \det J \, d\bar{x}$$

$$-2 \int_\Omega \sum_{i=1}^{3} L_i(V_{ij}) \int_0^{x_3} f_i(x_1, x_2, \rho) \det J(x_1, x_2, \rho) \, d\rho \, d\bar{x} . \tag{3.28}$$

Here, $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\Omega)^9$.

Proof. The computation of the directional derivative at an arbitrary point $\{U_{ij}\}_{i,j=\overline{1,3}} \in \mathcal{V}$ is similar to the deduction of the Euler equation (3.21). The last integral may be rewritten by using (3.19), (3.20), and the variant of (3.25)–(3.27) associated with (3.19), (3.20), in the following way:

$$\int\limits_{\Omega} \sum_{i=1}^{3} f_i \left[\mu_i + x_1\, M_i + x_2\, D_i\right] \det J \, d\bar{x}$$

$$= - \int\limits_{\Omega} \sum_{i=1}^{3} L_i(V_{ij}) \int\limits_{0}^{x_3} f_i(x_1, x_2, \rho) \det J(x_1, x_2, \rho)\, d\rho\, d\bar{x}\,,$$

as in the previous remark.

Remark: Using (3.28), one can solve (3.1)–(3.4) by standard gradient with projection methods.

We close this section with an abstract variant of problem (3.1)–(3.4). To this end, let $V \subset H$ be two separable Hilbert spaces with dense and continuous embedding, which are endowed with the scalar products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_H$, respectively. Let $A_1, A_2 : V \to V^*$ (V^* is the dual of V, while H is identified with its dual) be linear, continuous, symmetric, and positive definite operators.

We briefly comment on the equation

$$(A_1 + A_2)\, y = f \in H\,. \tag{3.29}$$

Under the above assumptions, equation (3.29) has a unique solution $y \in V$. We associate with it the optimal control problem

$$\text{Min } \left\{ \frac{1}{2}|w|_H^2 + \frac{1}{2}(A_2\, y, y)_{V^*\times V} \right\}, \tag{3.30}$$

$$A_1^{1/2}\, y = g + w\,, \tag{3.31}$$

where g is the unique solution of $A_1^{1/2}\, g = f$ and $A_1^{1/2} : V \to H$ is the square root of A_1, that is,

$$(A_1^{1/2}\, y\,,\, A_1^{1/2}\, v)_H = (A_1\, y, v)_{V^*\times V} \text{ for all } y, v \in V\,.$$

Clearly, $A_1^{1/2}$ is symmetric and positive definite, and equation (3.31) has a unique solution for any $w \in H$. Moreover, since the cost functional (3.30) is coercive and strictly convex, it is well known that the control problem (3.30), (3.31) has a unique optimal pair $[y^*, w^*] \in V \times H$.

We now take arbitrary variations of the form

$$y^* + \lambda\, z\,,\ \lambda \in \mathbb{R}\,,\ z \in V\,, \text{ and } w^* + \lambda\, v\,,\ v = A_1^{1/2}\, z\,.$$

Then the same argument as for (3.21) gives the Euler equation associated with (3.30), (3.31):

$$(w^*, v)_H + (A_2\, y^*, z)_{V^*\times V} = 0\,. \tag{3.32}$$

In (3.32), we replace

$$v = A_1^{1/2} z, \text{ and } w^* = A_1^{1/2} y^* - g = A_1^{1/2} y^* - A_1^{-1/2} f$$

to obtain that

$$\begin{aligned} 0 &= (A_2 y^*, z)_{V^* \times V} + (A_1^{1/2} y^* - A_1^{-1/2} f, A_1^{1/2} z)_H \\ &= (A_2 y^*, z)_{V^* \times V} + (A_1 y^*, z)_{V^* \times V} - (f, z)_{V^* \times V}, \end{aligned}$$

for any $z \in V$. This shows that y^* solves (3.29), which is a result that is similar to Theorem 3.3.

Remark: Notice that the solution to (3.30), (3.31) does not require the inversion of A_2; that is, in solving (3.29), we may separate the "good" part of A_1 of the differential operator and work just with it.

Remark: One may use (3.30), (3.31) directly in connection to (2.7) with $V = H_0^1(0, L)^9$ and $H = L^2(0, L)^9$, which is an alternative choice to (3.1)–(3.4). However, the construction of the square root of an operator may be a difficult task, so we do not pursue this idea, here.

References

[1] V. Arnăutu, H. Langmach, J. Sprekels, D. Tiba: *On the approximation and the optimization of plates*. Numer. Funct. Anal. Optim. **21:3–4** (2000), pp. 337–354.

[2] V. Arnăutu, J. Sprekels, D. Tiba: *Optimization problems for curved mechanical structures*. SIAM J. Control. Optim. **44:2** (2005), pp. 743–775.

[3] V. Arnăutu, J. Sprekels, D. Tiba: *A reduction approximation method for curved rods*. Numer. Funct. Anal. Optim. **26:2** (2005), pp. 139–155.

[4] Ph. Ciarlet: *Mathematical Elasticity. Vol. III. Theory of Shells*. North-Holland, Amsterdam 2000.

[5] A. Ignat, J. Sprekels, D. Tiba: *A model of a general elastic curved rod*. Math. Methods Appl. Sci. **25:10** (2002), pp. 835–854.

[6] F. Murat, A. Sili: *Comportement asymptotique des solutions du système de l'élasticité linéarisée anisotrope hétérogène dans des cylindres minces*. C. R. Acad. Sci. Paris, Sér. I Math. **328:2** (1999), pp. 179–184.

[7] P. Neittaanmäki, J. Sprekels, D. Tiba: *Optimization of Elliptic Systems. Theory and Applications*. Springer, New York 2006.

[8] O. Pironneau, R. Glowinski: *Sur une méthode quasi-directe pour l'opérateur biharmonique et ses applications à la résolution des équations de Navier–Stokes*. Ann. Sci. Math. Quebec **1:2** (1977), pp. 231–245.

[9] O. Pironneau, R. Glowinski: *Sur la résolution numérique du problème de Dirichlet pour l'opérateur biharmonique par une méthode quasi-directe*. C. R. Acad. Sci. Paris, Sér. A–B **282:4** (1976), pp. A223–A226.

[10] J. Sprekels, D. Tiba: *Sur les arches lipschitziennes*, C. R. Acad. Sci. Paris, Sér. I Math. **331:2** (2000), pp. 179–184.

[11] J. Sprekels, D. Tiba: *Control variational methods for differential equations*. In: Internat. Ser. Numer. Math. **139**, pp. 245–257. Birkhäuser, Basel 2002.

[12] J. Sprekels, D. Tiba: *An analytic approach to a generalized Naghdi shell model*, Adv. Math. Sci. Appl. **12:1** (2002), pp. 175–190.

[13] J. Sprekels, D. Tiba: *Optimal design of mechanical structures*. In LNPAM **242**, pp. 259–271. Chapman & Hall/CRC, Boca Raton, Fl 2005.

[14] D. Tiba, R. Vodak: *A general asymptotic model for Lipschitzian curved rods*. Adv. Math. Sci. Appl. **15:1** (2005), pp.137–198.